Constructions of Strict Lyapunov Functions: Stability, Robustness, Delays, and State Constraints

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Proposal for MTNS 2016 Mini-Course

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1 Motivation

The construction of strict Lyapunov functions is important for proving stability and robustness properties for nonlinear control systems. In some cases, stabilization problems can be solved with the help of nonstrict Lyapunov functions, which are proper and positive definite functions whose time derivatives are nonpositive along all solutions of the closed loop system. By properness and positive definiteness of a function V, we mean that V is zero at the equilibrium, positive at all other states, and satisfies $V(x) \to \infty$ as $|x| \to \infty$ or as x approaches the boundary of the state space. However, nonstrict Lyapunov functions by themselves are insufficient to solve asymptotic stabilization problems, since they do not ensure convergence to the equilibrium. Instead, one often uses nonstrict Lyapunov functions, combined with LaSalle invariance or a Matrosov approach [6].

However, even if one uses LaSalle invariance or standard Matrosov approaches, there is usually no guarantee of robustness, e.g., under control or model uncertainty. This helped motivate the lecturers' 'strictification' approach for converting nonstrict Lyapunov functions into strict ones [2]. A strict Lyapunov function is a proper positive definite function whose time derivative is negative along all solutions of the system outside the equilibrium. Strict Lyapunov functions also allow us to robustify controls, e.g., to prove robustness in the key sense of input-to-state stability (or ISS).

To see how this 'robustification' approach can be done in the special case of time invariant nonlinear control affine systems of the form $\dot{x} = f(x) + g(x)u(x)$, assume that we found a control $u(x)$ such that the closed loop system is globally asymptotically stable to 0, and that we have a strict Lyapunov function V for the closed loop system such that $-\dot{V}(x)$ is proper and positive definite, or equivalently, there is a class $\mathcal{K}_\infty$ function α such that $\dot{V}(x) \leq -\alpha(|x|)$ holds along all trajectories of the closed loop system [2]. Then the closed loop system $\dot{x} = f(x) + g(x)(u^\sharp(x) + \delta)$ has the ISS property with respect to the set of all measurable essentially bounded functions δ when we use $u^\sharp(x) = u(x) - (\nabla V(x)g(x))^\top$, i.e., we get ISS with respect to actuator errors δ [2].

However, to use $u^\sharp$, one needs formulas for the gradient ∇V of the strict Lyapunov function, and this was another motivation for the strictification approach, but there are other motivations. For instance, having closed form strict Lyapunov functions leads to explicit formulas for comparison functions in ISS estimates, and can make backstepping possible. Analogous results can be shown for systems with delays, where strict Lyapunov functions for undelayed systems are replaced by strict Lyapunov-Krasovskii functionals, whose domains are infinite dimensional sets of functions, and this makes it possible to quantify the effects of input delays on the control performance.

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2 Objectives

The preceding discussion motivates this mini-course, which presents several developments involving constructions of strict Lyapunov functions for systems with uncertainties, state constraints, or input delays. It will be understandable and beneficial to those who are familiar with the material in a standard graduate course on nonlinear control systems, and will consist of three 50-minute lectures.

Lecture 1: Matrosov's Approach (Lecturer: Frederic Mazenc)

The Matrosov approach involves combining known nonstrict Lyapunov functions for nonlinear systems with one or more so-called auxiliary functions, to build strict Lyapunov function for the systems [6]. While very general, it may not always be easy to find the auxiliary functions. In this lecture, we review a method for building the required auxiliary functions, based on iterated Lie derivatives, and we illustrate the method using a Lotka-Voltera dynamics [4]. We also provide an alternative Matrosov type construction for chains of exponential integrators where the right sides of the systems saturate, so they are not completely controllable [1]. This leads to stabilization and tracking results for a broad class of predator-prey and other time-varying biologically inspired control systems that have input delays or positivity constraints on the states of the systems.

Lecture 2: Lyapunov-Krasovskii Methods (Lecturer: Michael Malisoff)

Starting from a strict Lyapunov function V for a controlled system without delays, it is often possible to add integral terms to V to obtain a Lyapunov-Krasovskii functional for the corresponding input delayed system, and to find a range of admissible values for the delays that ensure robust asymptotic stability properties under control or model uncertainties. This contrasts with prediction or reduction methods that take the delay value into account in the control design. Although prediction and reduction often compensate for arbitrarily long delays, a potential advantage of the Lyapunov-Krasovskii approach is that it allows us to use simpler controllers that were originally designed for systems without delays. This lecture will discuss the Lyapunov-Krasovskii approach, including illustrations involving a key curve tracking dynamics that arises in marine robotics [5, 7].

Lecture 3: Robust Forward Invariance (Lecturer: Michael Malisoff)

Many applications such as collision avoidance problems require that tracking objectives be met, while also keeping the state in certain safe regions of the state space. Robust forward invariance can be used with Lyapunov-Krasovskii functionals to provide predictable tolerance and safety bounds that ensure that systems respect state constraints. It is a novel variant of the strong invariance property for differential inclusions, with the additional useful feature that it provides maximum allowable perturbation sets that the system can tolerate without violating the required tolerance and safety bounds. This lecture will discuss applications of robust forward invariance to human-pointer interactions under pointer acceleration [8], and to 3D curve tracking under delays and state constraints where an adaptive controller is used to identify unknown control gains [3].

3 Speaker Biographies

Frederic Mazenc was born in Cannes, France in 1969. He received his Ph.D. in Automatic Control and Mathematics from the CAS at Ecole des Mines de Paris in 1996. He was a Postdoctoral Fellow at CESAME at the University of Louvain in 1997. From 1998 to 1999, he was a Postdoctoral Fellow at the Centre for Process Systems Engineering at Imperial College. He was a CR at INRIA Lorraine from October 1999 to January 2004. In January 2004, he became CR1 at INRIA Sophia-Antipolis, and he has been CR1 at INRIA Saclay since January 2010. He received a best paper award from *IEEE Transactions on Control Systems Technology* at the 2006 IEEE Conference on Decision and Control, and a Best Presentation Award in an American Control Conference session. His current research interests include nonlinear control theory, differential equations with delay, robust control, and microbial ecology. He has more than 200 peer reviewed publications. Together with Michael Malisoff, he authored a research monograph entitled *Constructions of Strict Lyapunov Functions* in the Springer Communications and Control Engineering Series. He serves as Associate Editor for *Asian Journal of Control*, *European Journal of Control*, *IEEE Transactions on Automatic Control*, *Journal of Control and Decision*, and *Mathematical Control and Related Fields.*

Michael Malisoff was born in the City of New York and received his Ph.D. in 2000 from the Department of Mathematics at Rutgers University in New Brunswick, NJ. In 2000, he became a DARPA Research Associate at Washington University in Saint Louis, as part of the Joint Force Air Component Commander Project. In 2001, he joined the tenure track faculty of the Department of Mathematics at Louisiana State University in Baton Rouge, where he is currently the Roy Paul Daniels Professor #3 in the College of Science. His awards include the First Place Student Best Paper Award at the 1999 IEEE Conference on Decision and Control, two three-year US National Science Foundation Mathematical Sciences Priority Area grants, and 9 Best Presentation awards in American Control Conference sessions. He has more than 100 publications on optimal control, feedback design, and engineering applications. He has served as Associate Editor for *Automatica*, *IEEE Transactions on Automatic Control*, and *SIAM Journal on Control and Optimization.*

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